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VERTICAL MOTION OF A BODY OF REVOLUTION IN A STRATIFIED FLUID

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When a body moves in an inviscid fluid with a nonhomogeneous density, the effect of stratification on the hydrodynamic loads manifests itself through variable hydrostatic forces, additional forces caused by energy consumption for generation of internal gravity waves, and through the instantaneous response of the fluid to the external action, i.e., the response of the non-gravity fluid. Most theoretical results on motion of a body in an incompressible stratified fluid have been obtained within the framework of the linear theory of waves. The horizontal motion of a submerged body at constant velocity, when this problem is considered stationary, and small steady vibrations of a body with and without uniform horizontal motion have been studied most thoroughly. For motion of a body in an arbitrary direction in a fluid with a variable density gradient, one should solve a nonstationary problem. At present, the studies along this line have been primary devoted to the vertical motion of a body through the interface between two media and to the particular case of motion of a body to the free surface of a homogeneous fluid (see [1–8]). In some papers, the generation of internal waves localized near the interface is called transient radiation.

The difficulties of the exact solution of flow problems are often eliminated by an approximate simulation of a body by a system of mass sources and sinks taken from the theory of a homogeneous unbounded fluid.

In this paper, we consider the uniform vertical motion of a slender body of revolution with a density distribution in the form of a pycnocline. A sharp pycnocline is modeled by a two-layer fluid, and a smooth pycnocline is modeled by a three-layer fluid with an exponentially stratified middle layer and homogeneous upper and lower layers. The body begins to move at constant velocity far from the pycnocline, crosses it, and moves upward by a great distance. The problem is solved both with and without the Boussinesq approximation. In the latter case, the load occurring upon motion of the body in a non-gravity fluid is determined. The effects of body shape and velocity on buoyancy forces, and the effects of pycnocline thickness and density difference are also studied.

1. Field of a Point Source. The small motions of an initially undisturbed, incompressible, inviscid, stratified fluid in a uniform gravity field in the presence of a mass source with density $\rho_0 Q(\mathbf{x},t)$ in Cartesian coordinates $\mathbf{x} = (x, y, z)$ with the z axis directed vertically upward are described by the following system of linear equations:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p + \mathbf{F} = 0, \quad \frac{\partial \rho}{\partial t} + \rho_0'(z) w = 0, \quad \operatorname{div} \mathbf{u} = Q(\mathbf{x}, t). \tag{1.1}$$

Here $\mathbf{u} = (u, v, w)$, p, and ρ are the perturbations of the velocity vector, pressure, and density, $\rho_0(z)$ is the fluid density in the undisturbed state, $\mathbf{F} = (0, 0, g\rho)$ is the density vector of the mass forces, g is the acceleration of gravity, the prime denotes differentiation with respect to z, and t is time. The boundary and initial conditions are as follows: $\mathbf{u}, p \to 0$ ($|\mathbf{x}| \to \infty$) and $\mathbf{u} = \rho = 0$ ($t = 0$).

For an arbitrarily moving point source, we have $Q(\mathbf{x},t) = q(t)\delta(\mathbf{x} - \mathbf{Y}(t))$ [$q(t) \equiv 0$ for $t \leqslant 0$], where δ is the Dirac delta function, $\mathbf{Y}(t) = (y_1(t), y_2(t), y_3(t))$, and $\mathbf{x} = \mathbf{Y}(t)$ is the motion path of the source.

System (1.1) can be reduced to one equation for the vertical velocity component $w(\mathbf{x}, t)$:

$$\frac{\partial^2}{\partial t^2}\Big[\frac{\partial}{\partial z}\Big(\rho_0 \frac{\partial w}{\partial z}\Big) + \rho_0 \Delta_2 w\Big] + \rho_0 N^2 \Delta_2 w = \frac{\partial^3}{\partial t^2 \partial z}(\rho_0 Q). \tag{1.2}$$

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Here $N(z) = \sqrt{-g\rho_0'/\rho_0}$ is the buoyancy frequency and $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplace operator. The remaining quantities can be expressed in terms of w by means of the relations following from (1.1). In particular, the pressure calculated ignoring hydrostatic forces is equal to

$$p = \rho_0 \Delta_2^{-1} \frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial z} - Q\Big). \tag{1.3}$$

To determine the desired quantities, we perform a Fourier transform with respect to the horizontal variables x and y and time t:

$$w_*(\mu, \nu, z, \omega) = \int\limits_0^\infty e^{i\omega t}\, dt \int\limits_{-\infty}^{\infty} e^{-i\mu x}\, dx \int\limits_{-\infty}^{\infty} e^{-i\nu y} w(\mathrm{x}, t)\, dy.$$

Then, the Fourier transform of the vertical velocity w_* is described by the equation

$$(\rho_0 w_*')' - \rho_0 k^2\Big(1 - \frac{N^2}{\omega^2}\Big) w_* = (\rho_0 Q_*)', \quad k^2 = \mu^2 + \nu^2, \tag{1.4}$$

whose solution is expressed in terms of the Green function $G(k, z, \xi, \omega)$ subject to the equation

$$(\rho_0 G')' - \rho_0 k^2\Big(1 - \frac{N^2}{\omega^2}\Big) G = \delta(z - \xi) \tag{1.5}$$

and the boundary conditions $G \to 0$ ($|z| \to \infty$). Following [6], the function G can be represented as the sum of two terms: $G(k, z, \xi, \omega) = G_0(k, z, \xi) + G_1(k, z, \xi, \omega)$, where $G_0(k, z, \xi) = \lim\limits_{\omega\to\infty} G(k, z, \xi, \omega)$ does not depend on ω and is the solution of the equation

$$(\rho_0 G_0')' - \rho_0 k^2 G_0 = \delta(z - \xi), \quad G_0 \to 0 \quad (|z| \to \infty). \tag{1.6}$$

The function G_0 defines the instantaneous part of the fluid response to the external action and describes the part of the disturbance field that is carried away by the moving source and corresponds to the non-gravity fluid, i.e., to the zero vector of the mass forces $\mathbf{F}$ in (1.1). The remaining part $G_1(k, z, \xi, \omega)$ is a delayed response and describes the internal waves localized near the density variation levels. The function G_1 is a solution of the nonhomogeneous equation

$$(\rho_0 G_1')' - \rho_0 k^2\Big(1 - \frac{N^2}{\omega^2}\Big) G_1 = -\frac{k^2}{\omega^2} \rho_0 N^2 G_0, \tag{1.7}$$

and it vanishes as $|z| \to \infty$.

The solution of Eq. (1.7) can be represented as an expansion in the eigenfunctions of the following eigenvalue problem:

$$(\rho_0 W_n')' - \rho_0 k^2 (1 - N^2/\omega^2) W_n = 0, \quad W_n \to 0 \quad (|z| \to \infty). \tag{1.8}$$

The spectral properties of this problem have been well studied (see, for example, [9]). The spectrum $\omega_n^2(k)$ is positive and discrete with stable stratification $N(z) \geqslant 0$ over the entire range of depths and nonzero $N(z)$ only on a finite interval. The eigenfunctions $W_n(z)$ are orthogonal and normalized:

$$\int\limits_{-\infty}^{\infty} \rho_0(z) N^2(z) W_n^2(z)\, dz = 1. \tag{1.9}$$

As a result, for G_1 we have

$$G_1 = \sum_{n=1}^{\infty} F_n(\xi) W_n(z).$$

Here

$$F_n(\xi) = -\frac{\omega_n^2}{\omega_n^2 - (\omega + i\lambda)^2} \int\limits_{-\infty}^{\infty} \rho_0(\eta) N^2(\eta) G_0(\eta, \xi) W_n(\eta)\, d\eta,$$

where λ is a small positive constant.

Knowing the Green function, one can determine the Fourier transforms of the vertical velocity w_* and the pressure p_*, taking into account (1.3):

$$w_* = -\int_0^\infty q(\tau)\exp\left[i\omega\tau - i(\mu y_1(\tau) + \nu y_2(\tau))\right]\left[\rho_0(\xi)\frac{\partial G}{\partial \xi}\right]_{\xi=y_3(\tau)} d\tau,$$

$$p_* = -\frac{i\omega}{k^2}\rho_0(z)\int_0^\infty q(\tau)\exp\left[i\omega\tau - i(\mu y_1(\tau) + \nu y_2(\tau))\right]\left[\rho_0(\xi)\frac{\partial^2 G}{\partial z\partial \xi} + \delta(z-\xi)\right]_{\xi=y_3(\tau)} d\tau.$$

To the division of the Green function into two parts corresponds the following representation for the Fourier transform of the pressure: $p_* = p_*^0 + p_*^1$. In this case the first term p_*^0 describes the pressure perturbation in the non-gravity fluid and vanishes when the source is shut down:

$$p_*^0 = -\frac{i\omega}{k^2}\rho_0(z)\int_0^\infty q(\tau)\exp\left[i\omega\tau - i(\mu y_1(\tau) + \nu y_2(\tau))\right]M(z, y_3(\tau))d\tau \tag{1.10}$$

$$\left[M(z,\xi) = \rho_0(\xi)\frac{\partial^2 G_0}{\partial z\partial \xi} + \delta(z-\xi)\right].$$

The second term is the wave part of the pressure perturbation and does not vanish with shutdown of the source:

$$p_*^1 = -\frac{i\omega}{k^2}\rho_0(z)\int_0^\infty q(\tau)\exp\left[i\omega\tau - i(\mu y_1(\tau) + \nu y_2(\tau))\right]\left[\rho_0(\xi)\frac{\partial^2 G_1}{\partial z\partial \xi}\right]_{\xi=y_3(\tau)} d\tau.$$

Performing the inverse Fourier transforms

$$p = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega \int_{-\infty}^{\infty} e^{i\mu x}\, d\mu \int_{-\infty}^{\infty} e^{i\nu y} p_*\, d\nu$$

we have $p = p_0 + p_1$, where

$$p_0(\mathbf{x}, t) = \frac{\rho_0(z)}{2\pi}\frac{\partial}{\partial t}\left[q(t)\int_0^\infty \frac{dk}{k}J_0(kr(t))M(z, y_3(t))\right], \tag{1.11}$$

$$p_1(\mathbf{x}, t) = -\frac{\rho_0(z)}{2\pi}\int_0^t q(\tau)\rho_0(y_3(\tau))d\tau\int_0^\infty \frac{dk}{k}J_0(kr(\tau))\sum_{n=1}^\infty \omega_n^2\cos\omega_n(t-\tau)W_n'(z)\int_{-\infty}^\infty \rho_0(\eta)N^2(\eta)\frac{\partial G_0}{\partial \xi}\Big|_{\xi=y_3(\tau)} W_n(\eta)\,d\eta;$$

$r(\zeta) = [(x - y_1(\zeta))^2 + (y - y_2(\zeta))^2]^{1/2}$, and J_0 is a zero-order Bessel function of the first kind.

This solution can be somewhat simplified for a weakly stratified fluid by introduction of the Boussinesq approximation. In this approximation, the density difference from the constant value $\rho_s = \rho_0(0)$ in the equations of momentum conservation is taken into account only in the terms describing buoyancy, and, in inertial terms, the actual density is replaced by ρ_s.

In the Boussinesq approximation, Eq. (1.2) takes the form

$$\frac{\partial^2}{\partial t^2}\Delta w + N^2\Delta_2 w = \frac{\partial^3 Q}{\partial t^2\partial z},$$

where $N = \sqrt{-g\rho_0'/\rho_s}$ is the buoyancy frequency and Δ is a three-dimensional Laplace transform. In relations (1.3), $\rho_0(z)$ should be replaced by ρ_s. Similar changes should be performed in relations (1.4)–(1.9). In this case, Eq. (1.6) has a simple solution that corresponds to a homogeneous fluid: $G_0 = -e^{-k|z-\xi|}/(2\rho_s k)$. After integration in (1.11) we obtain the following result, which is well known for an infinite homogeneous fluid:

$$p_0(\mathbf{x}, t) = \frac{\rho_s}{4\pi}\frac{\partial}{\partial t}\left(\frac{q(t)}{r_1}\right) \qquad [r_1^2 = r^2(t) + (z - y_3(t))^2].$$

The simplest example of a stratified fluid is a two-layer fluid consisting of layers of different density. In the more general case, the sharp density jump between the layers can be replaced by a layer with continuously varying density (three-layer fluid). Below, we use the solutions obtained for theses models of fluid stratification.

2. Two-Layer Fluid. In the undisturbed state, the upper layer with density ρ_1 occupies the region $z > 0$, and the lower layer with density $\rho_2 = (1+\varepsilon)\rho_1$ $(\varepsilon > 0)$ occupies the region $z < 0$. The particular case of this fluid for $\varepsilon \to \infty$ is a semi-infinite homogeneous fluid with a free surface.

Using the results of [6], we describe briefly the solution of the problem in this case. The fluid density in the undisturbed state is conveniently written as $\rho_0(z) = \rho_s(1 - \gamma \operatorname{sgn} z)$, where $\rho_s = \rho_1(2+\varepsilon)/2$ and $\gamma = \varepsilon/(2+\varepsilon)$. The solution of Eq. (1.6) has the form

$$G_0(k,z,\xi) = -\frac{1}{2k\rho_0(\xi)}\Big[\mathrm{e}^{-k|z-\xi|} - \gamma \operatorname{sgn} \xi \mathrm{e}^{-k(|z|+|\xi|)}\Big].$$

The eigenvalue problem (1.8) has a unique solution, since a single wave mode exists in this fluid: $W_1 = \mathrm{e}^{-k|z|}/\sqrt{2\bar{g}\rho_s}$, $\omega_1 = \sqrt{\bar{g}k}$, and $\bar{g} = \gamma g$. Hence, the solutions for p_0 and p_1 in (1.11) are written as

$$p_0 = \frac{\rho_0(z)}{4\pi}\frac{\partial}{\partial t}\Big[q(t)\Big(\frac{1}{r_1} + \frac{\gamma \operatorname{sgn} z}{r_2}\Big)\Big],$$

$$p_1 = \frac{\bar{g}\rho_0(z)}{4\pi} \operatorname{sgn} z \int\limits_0^t q(\tau)[\operatorname{sgn}(y_3(\tau)) - \gamma]\,d\tau \int\limits_0^\infty kJ_0(kr)\mathrm{e}^{-k(|z|+|y_3(\tau)|)}\cos\omega_1(t-\tau)\,dk \qquad (2.1)$$

$$r_2^2 = r^2 + (|z| + |y_3(t)|)^2].$$

The velocity potentials in each layer with unsteady motion of a source in a two-layer fluid are determined by He et al. [10]. The pressure calculated from the velocity potentials coincides with that given in (2.1).

3. Three-Layer Fluid. In the undisturbed state, the continuous density distribution with constant values of ρ_1 in the upper layer $z > H$, with constant values of ρ_2 in the lower layer $z < -H$, and with an exponential variation in the middle layer with thickness $2H$ has the form

$$\rho_0(z) = \begin{cases} \rho_1 = \rho_s\mathrm{e}^{-\alpha H} & (z > H), \\ \rho_s\mathrm{e}^{-\alpha z} & (|z| < H), \\ \rho_2 = \rho_s\mathrm{e}^{\alpha H} & (z < -H). \end{cases} \qquad (3.1)$$

The buoyancy frequency is different from zero only in the middle layer, in which it has a value $N = \sqrt{\alpha g}$. For convenient comparison with the results for a two-layer fluid, we indicate the relation $\alpha = \ln(1+\varepsilon)/2H$.

In this case, Eq. (1.6) has constant coefficients, and its solution is conveniently written as

$$G_0 = \begin{cases} -w_1(z)w_2(\xi)/D & (z > \xi), \\ -w_1(\xi)w_2(z)/D & (z < \xi), \end{cases}$$

where $w_1(z)$ and $w_2(z)$ are two linearly independent solutions:

$$(w_1, w_2) = \begin{cases} \mathrm{e}^{-kz}, & \mathrm{e}^{2\alpha H}(b_2\mathrm{e}^{kz} - b_1\mathrm{e}^{-kz}) \quad (z > H), \\ a_1\mathrm{e}^{\gamma_1 z} + a_2\mathrm{e}^{\gamma_2 z}, & c_1\mathrm{e}^{\gamma_1 z} + c_2\mathrm{e}^{\gamma_2 z} \quad (|z| < H), \\ b_1\mathrm{e}^{kz} + b_2\mathrm{e}^{-kz}, & \mathrm{e}^{kz} \quad (z < -H). \end{cases}$$

Here $a_1 = -(k+\gamma_2)\mathrm{e}^{-H(\gamma_1+k)}/2\mu$, $a_2 = (\gamma_1 + k)\mathrm{e}^{-H(\gamma_2+k)}/2\mu$, $b_1 = \alpha(\mathrm{e}^{-2H\gamma_2} - \mathrm{e}^{-2H\gamma_1})/4\mu$, $b_2 = \mathrm{e}^{-2Hk}[(k+\mu)\mathrm{e}^{-2H\gamma_2} + (\mu-k)\mathrm{e}^{-2H\gamma_1}]/2\mu$, $c_1 = (k-\gamma_2)\mathrm{e}^{H(\gamma_1-k)}/2\mu$, $c_2 = (\gamma_1 - k)\mathrm{e}^{H(\gamma_2-k)}/2\mu$, $\gamma_1 = \alpha/2+\mu$, $\gamma_2 = \alpha/2-\mu$, and $\mu = \sqrt{k^2+\alpha^2/4}$.

The Wronskian $D = \rho_0(\xi)[w_1(\xi)w_2'(\xi) - w_1'(\xi)w_2(\xi)]$ does not depend on ξ:

$$D = \rho_s k\mathrm{e}^{2H(\mu-k)}[\mu + k + (\mu - k)\mathrm{e}^{-4\mu H}]/\mu.$$

The function $M(z,\xi)$ in (1.10) has nine different representations, according to the layer in which the variables

z and ξ are present: for $z > H$, we have

$$M(z,\xi) = \frac{1}{2b_2}\begin{cases} k[b_1 e^{-k(z+\xi)} + b_2 e^{-k|z-\xi|}] & (\xi > H),\\ e^{-(kz+\alpha H)}(c_1\gamma_1 e^{-\gamma_2\xi} + c_2\gamma_2 e^{-\gamma_1\xi}) & (|\xi| < H),\\ k e^{k(\xi - z)} & (\xi < -H),\end{cases}$$

for $|z| < H$,

$$M(z,\xi) = \frac{1}{2b_2}\begin{cases} e^{-(k\xi+2\alpha H)}(c_1\gamma_1 e^{\gamma_1 z} + c_2\gamma_2 e^{\gamma_2 z}) & (\xi > H),\\ -(e^{-\alpha H}/k)[a_1c_1\gamma_1^2 e^{\gamma_1 z - \gamma_2\xi} + a_2c_2\gamma_2^2 e^{\gamma_2 z - \gamma_1\xi} & \\ \qquad -k^2(a_2c_1 e^{\alpha(z-\xi)/2 - \mu|z-\xi|} + a_1c_2 e^{\alpha(z-\xi)/2+\mu|z-\xi|})] & (|\xi| < H),\\ -e^{k\xi}(a_1\gamma_1 e^{\gamma_1 z} + a_2\gamma_2 e^{\gamma_2 z}) & (\xi < -H),\end{cases}$$

and for $z < -H$,

$$M(z,\xi) = \frac{1}{2b_2}\begin{cases} k e^{k(z-\xi)-2\alpha H} & (\xi > H),\\ -e^{kz-\alpha H}(a_1\gamma_1 e^{-\gamma_2\xi} + a_2\gamma_2 e^{-\gamma_1\xi}) & (|\xi| < H),\\ k[b_2 e^{-k|z-\xi|} - b_1 e^{k(z+\xi)}] & (\xi < -H).\end{cases}$$

The solution of the eigenvalue problem (1.8) for this stratification is presented in detail in [9, 11]. We give briefly the main results. The countable system of eigenfunctions $W_n(z)$ has the form

$$W_n = \Lambda^{-1/2}\begin{cases} e^{-kz} & (z > H),\\ e^{\alpha z/2}(A\sin\mu_1 z + B\cos\mu_1 z) & (|z| < H),\\ Ce^{kz} & (z < -H),\end{cases} \tag{3.2}$$

where

$$A = \frac{e^{-kH}}{2\mu_1\cos\mu_1 H}\Big[C\sigma\Big(k - \frac{\alpha}{2}\Big) - \frac{1}{\sigma}\Big(k + \frac{\alpha}{2}\Big)\Big];\quad B = \frac{e^{-kH}}{2\cos\mu_1 H}\Big(C\sigma + \frac{1}{\sigma}\Big);$$

$$C = \sigma^{-2}\begin{cases} (k + \alpha/2 + \mu_1\mathrm{cotan}\mu_1 H)/(k - \alpha/2 + \mu_1\mathrm{cotan}\mu_1 H) & (\text{uneven } n),\\ (k + \alpha/2 - \mu_1\tan\mu_1 H)/(\alpha/2 - k + \mu_1\tan\mu_1 H) & (\text{even } n);\end{cases}$$

$\sigma = e^{\alpha H/2}$, $\mu_1 = \sqrt{k^2(N^2/\omega^2 - 1) - \alpha^2/4}$, and the normalization factor

$$\Lambda = \rho_s N^2\Big[(A^2 + B^2)H + \frac{B^2 - A^2}{2\mu_1}\sin 2\mu_1 H\Big].$$

The dispersion relations $\omega_n(k)$ are determined by solution of the equation

$$\tan 2\mu_1 H = 2\mu_1\omega^2/[k(N^2 - 2\omega^2)]. \tag{3.3}$$

Obviously, one of the solutions of this equation is the solution $\mu_1 = 0$, which corresponds to $\omega_0/N = 2k/\sqrt{4k^2 + \alpha^2}$ and is called the "zero mode" in [11]. It is possible that, it does not make a contribution to the required solution only for one special value of k. For $\omega < \omega_0$ the value of μ_1 becomes imaginary, and Eq. (3.3) has a unique solution, which describes the initial portion of the dispersion curve of the first mode for not large k. For $\omega > \omega_0$, the continuation of this curve is determined by the real values of μ_1.

From (3.3), we determine the behavior of dispersion relations in the limiting case of small k:

$$\omega_1 \approx \beta_1\sqrt{k},\qquad \omega_n \approx \beta_n k\quad (k \to 0)$$
$$\Big[\beta_1 = N\sqrt{\frac{\tanh\alpha H}{\alpha}},\quad \beta_n = \frac{2NH}{\sqrt{\alpha^2H^2 + \pi^2(n-1)^2}}\quad (n \geqslant 2)\Big]. \tag{3.4}$$

In analyzing this problem in the Boussinesq approximation, we replace the exponential density distribution in the middle layer of the undisturbed state of the fluid (3.1) by the following linear distribution in a weakly stratified fluid:

$$\rho_0(z) = \rho_s \begin{cases} 2/(2+\varepsilon) & (z > H), \\ 1 - \gamma z/H & (|z| < H), \\ 2(1+\varepsilon)/(2+\varepsilon) & (z < -H). \end{cases}$$

In this case, the buoyancy frequency is also constant in the middle layer ($N = \sqrt{\bar{g}/H}$). The expressions for the eigenfunctions and the dispersion equation can be obtained from (3.2) and (3.3) by setting $\alpha = 0$. Note that from the dispersion equation in the Boussinesq approximation one can determine the explicit dependence of the wavenumber on the frequency:

$$k_n(\omega) = \frac{1}{2H} \begin{cases} \dfrac{\omega}{\sqrt{N^2-\omega^2}}\Big[(n-1)\pi + \arctan\Big(\dfrac{2\omega\sqrt{N^2-\omega^2}}{N^2-2\omega^2}\Big)\Big] & (\omega < N/\sqrt{2}), \\ \pi(n-1/2) & (\omega = N/\sqrt{2}), \\ \dfrac{\omega}{\sqrt{N^2-\omega^2}}\Big[n\pi + \arctan\Big(\dfrac{2\omega\sqrt{N^2-\omega^2}}{N^2-2\omega^2}\Big)\Big] & (\omega > N/\sqrt{2}). \end{cases}$$

4. Slender Body of Revolution. In what follows we restrict ourselves to the case of vertical motion of a source of constant intensity at velocity U. It is convenient to perform a shift in time and assume that the moment the source intersects the coordinate origin is zero time. Then, the motion trajectory of the source is written as $y_1(t) = y_2(t) = 0$, $y_3 = Ut$, and $r^2 = x^2 + y^2$, and the fluid flow becomes axisymmetric.

As is known, motion of a slender body of revolution along its own axis in an infinite homogeneous fluid can be modeled by motion of a system of point singularities located continuously on the body axis (see, for example, [2]). Suppose in the moving coordinate system $x_1 = x$, $y_1 = y$, and $z_1 = z - Ut$, the equation of body surface has the form $r = f(z_1)$. The function $f(z_1)$ is assumed to be even and bounded together with the first derivative everywhere except in small vicinities of the end points, at which the derivative can have singularities. Let a and b denote the half-width and half-length, respectively, of a slender body, $a/b \ll 1$.

The system of point singularities equivalent to this body has the following distribution on the z_1 axis in the interval $|z_1| \leqslant b$: $q(z_1, t) = -2\pi U f(z_1) f'(z_1) = -US'(z_1)$, where $S = \pi f^2$ is the cross-sectional area of the body.

Obviously, in a stratified fluid, this approximation models a body with a time-varying shape. It is assumed, however, that for weak stratification these variations are small.

The total fluid pressure P caused by motion of this system of singularities is written as

$$P(r, z, t) = \int_{-b}^{b} q(s, t)\, p(r, z, t, s)\, ds.$$

In this case, in relations (1.11) for p, one should set $q(\zeta) = 1$, $y_1 = y_2 = 0$, and $y_3(\zeta) = s + U\zeta$.

After integration of the pressure over the surface, the vertical force R acting on the slender body is defined by

$$R(t) = \int_{-b}^{b} P(f(\eta), \eta + Ut, t) S'(\eta)\, d\eta.$$

Without introduction of the Boussinesq approximation, the vertical force is the sum of two terms. $R = R_0 + R_1$, the first of which corresponds to motion of the body in a non-gravity fluid, and the second describes wave action.

In the Boussinesq approximation, the vertical force is determined only by the wave component of the pressure p_1 of the point source, since, in an ideal infinite homogeneous fluid, the body drag for translation motion is equal to zero (d'Alembert's paradox).

The final expressions for the vertical force acting on a slender body moving in a two-layer fluid have the form

$$R_0 = -\frac{U^2}{4\pi}\Big\{ \int_{-b}^{b} \rho_0(\eta+Ut)S'(\eta)d\eta \int_{-b}^{b} \frac{(\eta-\zeta)S'(\zeta)d\zeta}{[f^2(\eta)+(\eta-\zeta)^2]^{3/2}} - \gamma \int_{-b}^{b} \rho(\eta+Ut)\mathrm{sgn}\,(\eta+Ut)S'(\eta)d\eta$$

$$\times \int_{-b}^{b} \frac{(|\eta+Ut|+|\zeta+Ut|)}{[f^2(\eta)+(|\eta+Ut|+|\zeta+Ut|)^2]^{3/2}}\ \mathrm{sgn}\ (\zeta+Ut)S'(\zeta)\,d\zeta\Big\}; \tag{4.1}$$

$$R_1 = -\frac{\bar{g}U}{4\pi}\int_{-b}^{b} \rho_0(Ut+\eta)\ \mathrm{sgn}\ (\eta+Ut)S'(\eta)\,d\eta \int_{-b}^{b} S'(\zeta)\,d\zeta$$

$$\times \int_{-\infty}^{t} [\mathrm{sgn}(\zeta+U\tau)-\gamma]\,d\tau \int_{0}^{\infty} kJ_0(kf(\eta))\mathrm{e}^{-k(|\eta+Ut|+|\zeta+U\tau|)}\cos\omega_1(t-\tau)\,dk. \tag{4.2}$$

In a three-layer fluid without Boussinesq approximation,

$$R_0 = -\frac{U^2}{2\pi}\int_{-b}^{b} \rho_0(\eta+Ut)S'(\eta)\,d\eta \int_{-b}^{b} S'(\zeta)\,d\zeta \int_{0}^{\infty} \frac{dk}{k} J_0(kf(\eta))M_1(\eta+Ut,\zeta+Ut)\,dk; \tag{4.3}$$

$$R_1 = \frac{UN^2}{2\pi}\sum_{n=1}^{\infty}\int_{-b}^{b} \rho_0(\eta+Ut)S'(\eta)\,d\eta \int_{-b}^{b} S'(\zeta)\,d\zeta \int_{-\infty}^{t} \rho_0(\zeta+U\tau)\,d\tau$$

$$\times \int_{0}^{\infty} \frac{dk}{k} J_0(kf(\eta))\omega_n^2\Phi_n(\zeta+U\tau)W_n'(\eta+Ut)\cos\omega_n(t-\tau) \tag{4.4}$$

$$\Big[M_1(z,\xi) = \frac{\partial M(z,\xi)}{\partial\xi}, \quad \Phi_n(\xi) = \int_{-H}^{H} \rho_0(s)\frac{\partial G_0(s,\xi)}{\partial \xi} W_n(s)\,ds\Big].$$

In the Boussinesq approximation, relation (4.4) for the wave component of the vertical force is somewhat simplified:

$$R_1 = -\frac{\rho_s^2 UN^2}{4\pi}\sum_{n=1}^{\infty}\int_{-b}^{b} S'(\eta)\,d\eta \int_{-b}^{b} S'(\zeta)\,d\zeta \int_{-\infty}^{t} d\tau \int_{0}^{\infty} \frac{J_0(kf(\eta))}{k}\omega_n^2\Psi_n(\zeta+U\tau)W_n'(\eta+Ut)$$

$$\times \cos\omega_n(t-\tau)\,dk \qquad \Big[\Psi_n(\xi) = \int_{-H}^{H} \mathrm{e}^{-k|s-\xi|}\ \mathrm{sgn}\ (s-\xi)W_n(s)\,ds\Big]. \tag{4.5}$$

Note that [12] the energy losses due to the radiation of internal gravity waves by a given point source of mass was determined in [12] using the Boussinesq approximation. From these results we can also calculate the vertical fluid response caused by a distributed source, and for the problem considered we obtain

$$R_1 = -\frac{\rho_s^2 U}{2\pi}\sum_{n=1}^{\infty}\int_{-b}^{b} S'(\eta)\,d\eta \int_{-b}^{b} S'(\zeta)\,d\zeta \int_{-\infty}^{t} d\tau \int_{0}^{\infty} \frac{\omega_n^4}{k^3} W_n'(\zeta+U\tau)W_n'(\eta+Ut)\cos\omega_n(t-\tau)\,dk. \tag{4.6}$$

For a slender body, the Bessel function in (4.4) and (4.5) can be considered approximately equal to 1. The numerical calculations below are performed in the same approximation. It is not hard to show that, for the fluid stratification studied, expressions (4.5) and (4.6) are identically equal.

5. Numerical Results. For specific calculations we chose bodies of three shapes with different degrees

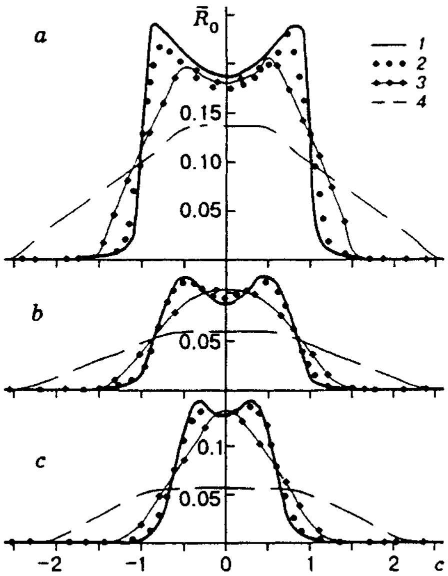

Fig. 1

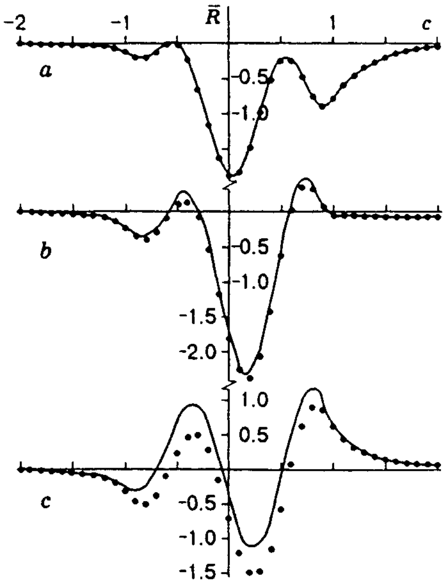

Fig. 2

of sharpness at the end points:

$$f(z_1) = a \begin{cases} \sqrt{1 - z_1^2/b^2} & \text{(B1)}, \\ \cos(\pi z_1/2b) & \text{(B2)}, \\ (1 + \cos(\pi z_1/b))/2 & \text{(B3)}. \end{cases} \tag{5.1}$$

Spheroid B1 is a blunt body, body B2 has an acute apex angle, and body B3 has zero apex angle. The volumes of these bodies at constant values of a and b are related to one another as $1:3/4:9/16$. In the calculations we assumed that $a/b = 0.1$.

The choice of body shape in the form of (5.1) yields simple density distribution models. In addition it allows one to perform analytically some integrations in (4.2), (4.4), and (4.5) and reduce these expressions to double integrals: with respect to η and k in (4.2) and with respect to k and τ in (4.4) and (4.5).

The vertical force $\bar{R}_0 = R_0 b^2/(\rho_s a^4 U^2)$ calculated by integration of (4.1) and (4.3) is shown as a function of $c = Ut/b$ in Figs. 1a–c for bodies B1, B2, and B3, respectively, in a two- or three-layer fluid for $\varepsilon = 0.03$. Curves 1 correspond to a two-layer fluid, and curves 2–4 to a three-layer fluid with thicknesses of the middle layer $H/b = 0.2$, 0.5, and 1.5. As noted by Porfir'ev [5], in studies of motion of a body to a free surface of a homogeneous fluid, the vertical force in a non-gravity fluid is always directed upward. Exact use of (4.1) for a blunt body B1 always leads to an infinite value of R_0 for $|Ut/b| = 1$ (see, for example, [5]). To eliminate this feature in the calculations for body B1, the mass sources were distributed not over the entire interval $|z_1| \leqslant b$, but only between the foci of the ellipse $|z_1| \leqslant \sqrt{b^2 - a^2}$, by analogy with the well-known solution on modeling a spheroid in the axial flow of a distribution of dipoles [13]. Obviously, for a thin stratified layer, the behavior of the vertical force is similar to that in the case of a two-layer fluid. With increase in the pycnocline thickness, the maximum value of the vertical force decreases.

A comparison of the total vertical force $\bar{R} = Rb/(\rho_s \bar{g} a^4)$ determined from (4.1) and (4.2) with that arising in a non-gravity fluid for body B2 is shown in Figs. 2 and 3 for $\varepsilon = 0.03$ and 0.3, respectively. Figures 2a–c, and 3a–c are given for Froude numbers $\mathrm{Fr} = U/\sqrt{\bar{g}b} = 0.5$, 1, and 2. The solid curves in Figs. 2 and 3 show the total force $\bar{R}$, and the dashed curve shows an analogous dimensionless complex for the wave component of the vertical force. The total work expended for wave generation in the vertical motion of a body in a two-layer fluid is determined by Warren [2]. According to [2], for bodies of various shapes, the wave losses are maximal for $\mathrm{Fr} \sim 1$. These conclusions are supported by comparison of the maximum values of the wave

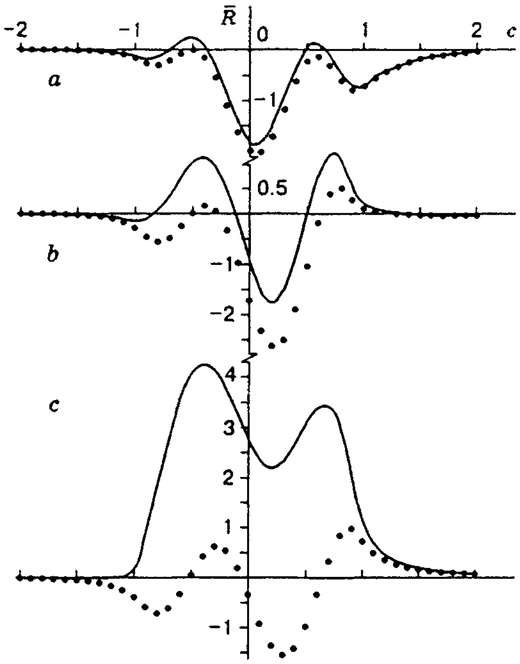

Fig. 3

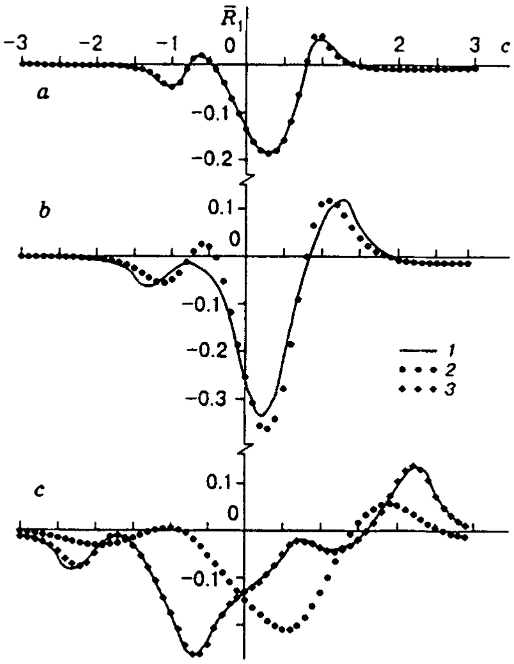

Fig. 4

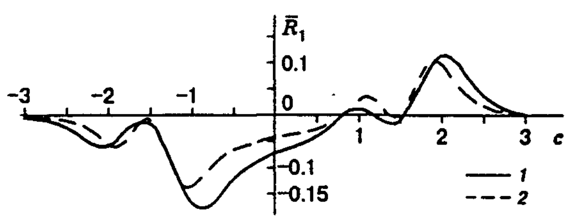

Fig. 5

component of the force in Figs. 2 and 3. The body drag in a non-gravity fluid is proportional to Fr^2. Hence, it can be concluded that, for high body velocities, the weight of the fluid has an insignificant effect, and, for $\mathrm{Fr} \sim 1$, it dominates. For low velocities ($\mathrm{Fr} \ll 1$), the effect of both factors are small. A comparison of Figs. 2 and 3 shows that the effects of the density difference in the dimensionless variables used is most marked for the non-gravity component of the vertical force, and the wave component changes only slightly.

The wave load for a three-layer fluid, $\bar{R}_1 = R_1/(\rho_s a^4 N^2)$, calculated in the Boussinesq approximation using (4.5) or, what is the same, (4.6) is shown in Fig. 4a–c for body B1 for $\varepsilon = 0.03$, $\mathrm{Fr} = 1$ and thicknesses of the middle layer $H/b = 0.2$, 0.5, and 1.5, respectively. In this problem,the contribution of various modes is of interest. In the calculations, 20 modes were considered (curves 1). In the case of a thin pycnocline ($H/b = 0.2$), the total load is determined primarily by the first mode (curves 2), and the contribution of the subsequent modes is small. With an increase in the pycnocline thickness, the contributions of higher modes increase. Figure 4c shows, along with the contributions of the first mode, the sum of the first five modes (curve 3). Evidently, the wave load for the pycnocline thicknesses considered is determined primarily by lower modes.

Figure 5 shows the effect of the body shape on the wave component of the vertical force calculated for $\varepsilon = 0.03$, $H/b = 1.5$, and $\mathrm{Fr} = 1$ in the Boussinesq approximation. Curves 1 and 2 correspond to bodies B2 and B3, respectively (similar results for body B1 are shown in Fig. 4c). Evidently, the maxima of the magnitudes of the wave force are approximately proportional to the volumes of these bodies.

Determination of the wave component of the vertical load without introducing the Boussinesq approximation is quite laborious. However, the results of the numerical integration of (4.4) indicate that the wave loads determined with and without the Boussinesq approximation practically coincide for relatively small values of the density difference between the upper and lower layers ($\varepsilon \leqslant 0.3$). This fact could have been predicted by comparing the maximum phase velocities of the internal wave in the Boussinesq approximation and without it. According to (3.4), the relative difference between the maximum phase velocities does not exceed 1% in the indicated range of ε. The difference between the wave forces calculated using (4.4) and (4.5) did not exceed the indicated value.

The numerical results presented allow one to estimate the effect of a smooth pycnocline on various components of the vertical force acting on a moving body.

The results considered can be used in studies of body motion in an arbitrary path in a stratified fluid, and also in developing a numerical method of boundary elements for determining the pressure on the body surface with exact satisfaction of the nonpenetration condition.

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